

Padé expansion and nucleon-nucleon scattering in coupled channels

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We extend our Padé-aided analysis of the nonperturbative renormalization of nucleon-nucleon scattering to the case of coupled channels.

Since the significant effective field theory (EFT) approach to nucleon systems[1, 2], there has been creating controversies about the consistent renormalization and EFT power counting in nonperturbative regime[3, 4]. The main difficulty is due to the nontrivial prescription dependence developed in nonperturbative regime, where some wisdoms about renormalization established within perturbative regimes cease to apply directly[5]. Therefore, the nonperturbative prescription dependence must be removed through imposing appropriate boundary conditions which are usually implemented through various forms of data fitting either explicitly or implicitly[6, 7, 8, 9, 10, 11, 12, 13]. In more recent literature, nonperturbative counter terms, or equivalently, nonperturbative parametrization of the renormalization prescription has been the main focus[14, 15]. That is, due to the difficulty in treating the issues in nonperturbative regimes, the key issue is to find more efficient parametrization of nonperturbative prescription.

In this regard, we have performed an analysis and treatment of the nonperturbative prescription dependence basing on Padé approximant of an important nonperturbative factor of T -matrix for nucleon-nucleon (NN) scattering in uncoupled channels[16, 17, 18]. In this report, we will extend our analysis to coupled channels. Some general theoretical and technical issues associated with coupled channels will be addressed first, then we will illustrate our method in 3D_3 - 3G_3 .

The object under consideration is the T -matrix for nucleon-nucleon scattering processes at low energies. According to Weinberg's proposal[19], this T -matrix should be solved from Lippmann-Schwinger (LS) equation with the potential to be systematically constructed using χ PT as the low energy effective theory of QCD. For coupled channels, such LSE's read,

$$\begin{aligned} \mathbf{T}(p', p; E) &= \mathbf{V}(p', p; E) + \int_k G_0(k; E^+) \mathbf{V}(p', k; E) \times \mathbf{T}(k, p; E), \\ G_0(k; E^+) &\equiv \frac{1}{E^+ - k^2/M}, \quad E^+ \equiv E + i\epsilon, \end{aligned} \quad (1)$$

with E being the nucleon energy in the center mass frame, M the nucleon mass, $p' = |\mathbf{p}'|$, $p = |\mathbf{p}|$. The bold-faced capital letters represent the 2×2 matrix-valued objects in the angular quantum number space. The convolution is understood as already regularized and/or renormalized in an unspecified prescription in order to make our discussions generally valid. Following Refs.[16, 17, 18], the above LSE for coupled channels could be transformed into a compact and hence nonperturbative parametrization of T -matrix as below,

$$\mathbf{T}^{-1}(p', p; E) = \mathbf{V}^{-1}(p', p; E) - \mathcal{G}(p', p; E), \quad (2)$$

$$\mathcal{G}(p', p; E) \equiv \mathbf{V}^{-1}(p', p; E) \times \left[\int_k G_0(k; E^+) \mathbf{V}(p', k; E) \times \mathbf{T}(k, p; E) \right] \times \mathbf{T}^{-1}(p', p; E) \quad (3)$$

where the factor \mathcal{G} assumes all the 'loop' processes generated by \mathbf{V} in the field-theoretical terminology. Making use of the K -matrix formalism, the unitarity of such compact T -matrices follows immediately[16, 17, 18]. It is also easy to verify the inverse relation in the coupled channels:

$$\mathbf{T} \times \mathbf{T}^{-1} = \left(\mathbf{V} + \int G_0 \mathbf{V} \times \mathbf{T} \right) \times \mathbf{T}^{-1} = \mathbf{V} \times \mathbf{T}^{-1} + \mathbf{V} \times \mathcal{G} = \mathbf{V} \times (\mathbf{T}^{-1} + \mathcal{G}) = \mathbf{V} \times \mathbf{V}^{-1} = \mathbf{I}, \quad (4)$$

$$\mathbf{T}^{-1} \times \mathbf{T} = (\mathbf{V}^{-1} - \mathcal{G}) \times \mathbf{T} = \mathbf{V}^{-1} \times \mathbf{T} - \mathcal{G} \times \mathbf{T} = \mathbf{V}^{-1} \times \left(\mathbf{T} - \int G_0 \mathbf{V} \times \mathbf{T} \right) = \mathbf{V}^{-1} \times \mathbf{V} = \mathbf{I}, \quad (5)$$

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with \mathbf{I} denoting the 2×2 unit matrix. Obviously, in terms of \mathbf{V} and \mathcal{G} , \mathbf{T} are nonperturbative objects.

It is interesting to note that in terms of the standard parametrization of S -matrix,

$$\mathbf{S} = \begin{pmatrix} \cos 2\epsilon_j(p) \exp[2i\delta_{j-1}^{1j}(p)] & i \sin 2\epsilon_j(p) \exp[i(\delta_{j-1}^{1j}(p) + \delta_{j+1}^{1j}(p))] \\ i \sin 2\epsilon_j(p) \exp[i(\delta_{j+1}^{1j}(p) + \delta_{j-1}^{1j}(p))] & \cos 2\epsilon_j(p) \exp[2i\delta_{j+1}^{1j}(p)] \end{pmatrix} = \mathbf{I} - i \frac{Mp}{2\pi} \mathbf{T}, \quad (6)$$

we could find that,

$$\mathbf{T}^{-1} = i \frac{Mp}{4\pi} \mathbf{I} + \frac{Mp}{4\pi} \begin{pmatrix} \frac{\sin(\delta_{j-1}^{1j} + \delta_{j+1}^{1j}) - \sin(\delta_{j-1}^{1j} - \delta_{j+1}^{1j}) \cos(2\epsilon_j)}{\cos(\delta_{j-1}^{1j} + \delta_{j+1}^{1j}) - \cos(\delta_{j-1}^{1j} - \delta_{j+1}^{1j}) \cos(2\epsilon_j)}, & \frac{-\sin(2\epsilon_j)}{\cos(\delta_{j-1}^{1j} + \delta_{j+1}^{1j}) - \cos(\delta_{j-1}^{1j} - \delta_{j+1}^{1j}) \cos(2\epsilon_j)} \\ \frac{-\sin(2\epsilon_j)}{\cos(\delta_{j-1}^{1j} + \delta_{j+1}^{1j}) - \cos(\delta_{j-1}^{1j} - \delta_{j+1}^{1j}) \cos(2\epsilon_j)}, & \frac{\sin(\delta_{j-1}^{1j} + \delta_{j+1}^{1j}) - \sin(\delta_{j-1}^{1j} - \delta_{j+1}^{1j}) \cos(2\epsilon_j)}{\cos(\delta_{j-1}^{1j} + \delta_{j+1}^{1j}) - \cos(\delta_{j-1}^{1j} - \delta_{j+1}^{1j}) \cos(2\epsilon_j)} \end{pmatrix}. \quad (7)$$

Thus, the imaginary part of \mathbf{T}^{-1} is simple and proportional to a unit matrix, the off-diagonal entries of \mathbf{T}^{-1} are real numbers only. The unitarity now reads: $\mathbf{T}^{-1} - (\mathbf{T}^{-1})^* = i \frac{Mp}{2\pi} \mathbf{I}$. Only the real part of \mathbf{T}^{-1} (or \mathcal{G}) is subject to nonperturbative renormalization and hence the unitarity is not affected by renormalization. The prescription dependence is exclusively contained in the real part of the factor \mathcal{G} .

The motivations and plausibility of employing Padé approximant to \mathcal{G} were already demonstrated in Refs.[16, 17, 18]. In this approximation, the unitarity of T -matrices is automatically preserved, a virtue that is welcome in hadron physics[20]. Another important virtue is its generality and flexibility in parametrizing the prescription dependence, avoiding being stuck in or confined to a special prescription that might not be quite compatible with physical boundaries.

Now it is clear that \mathcal{G} is a 2×2 matrix with the diagonal entries being complex, while the off-diagonal ones real. The Padé approximant is applied to the real part of any matrix element, diagonal or off-diagonal ($p = \sqrt{ME}$):

$$\text{Re}\{\mathcal{G}(p)_{l+\Delta, l+\Delta'}\} |_{\text{Padé}} = \frac{\sum_k N_{l+\Delta, l+\Delta'; k} p^{2k}}{\sum_{k'} D_{l+\Delta, l+\Delta'; k'} p^{2k'}}, \quad \Delta, \Delta' = 0, 2. \quad (8)$$

Evidently, in Padé approximant, the prescription dependence is contained the parameters $[N_{\dots; k}, D_{\dots; k'}]$. In other words, $[N_{\dots; k}, D_{\dots; k'}]$ serve as approximate parametrization of the nonperturbative prescription.

Then the renormalized \mathbf{T} could be approximately parametrized in nonperturbative regime as follows,

$$\mathbf{T}_{\text{Padé}}^{-1}(p; [g_{\dots}, C_{\dots}]; [N_{\dots}, D_{\dots}]) = \mathbf{V}^{-1}(p; [g_{\dots}, C_{\dots}]) - \left[\frac{\sum_k N_{l+\Delta, l+\Delta'; k} p^{2k}}{\sum_{k'} D_{l+\Delta, l+\Delta'; k'} p^{2k'}} \right]_{2 \times 2} + i \frac{Mp}{4\pi} \mathbf{I}. \quad (9)$$

Simple and coarse as it is, such nonperturbative parametrization of the T -matrix contains all contributing parameters: EFT couplings $[g_{\dots}, C_{\dots}]$, and renormalization prescription parameterized in terms of $[N_{\dots; k}, D_{\dots; k'}]$. As a byproduct, the Padé parameters allows us in principle to effectively imitate any renormalization prescription of \mathbf{T} through corresponding definition of $[N_{\dots; k}, D_{\dots; k'}]$.

For our approximation to be sensible, the Padé parameters must be appropriately determined. Then their magnitude orders should be in accordance with EFT power counting. For a general EFT power counting, one may expect that:

$$\left\{ \frac{N_{\dots; i}}{N_{\dots; 0}}, \frac{D_{\dots; i}}{D_{\dots; 0}} \right\} \sim \Lambda^{F(i)} \mu^{f(i)}, \quad (10)$$

with $\Lambda (\sim 500\text{MeV})$ being the upper EFT scale, μ the typical EFT scale, here, say, $\sim (10, 100)\text{MeV}$. F, f are some counting functions. Note that the reflection of EFT power counting in the factor \mathcal{G} is completely nonperturbative, in sheer contrast to the conventional understandings.

Occasional large deviation from such rules should be due to unnatural behaviors of the NN scattering. The EFT approach would be indeed problematic only if no power counting scheme could be sensibly realized in any renormalization prescription. In other words, the failure of some power counting schemes does not imply the very failure of the EFT approach.

In general, the intrinsic scales involved in the Padé parameters should be Λ and μ :

$$\left| \frac{N_{\dots; 0}}{N_{\dots; k}} \right|^{\frac{1}{2k}} \text{ or } \left| \frac{D_{\dots; 0}}{D_{\dots; k}} \right|^{\frac{1}{2k}} \sim \mu^\alpha \Lambda^{1-\alpha}, \quad \alpha \in (0, 1.0), \quad k > 0. \quad (11)$$

As $\dim[\mathcal{G}] = 2$, we choose $\dim[D_{\dots;0}] = 2$ and hence $|D_{\dots;0}|^{\frac{1}{2}} \sim \mu^\alpha \Lambda^{1-\alpha}$, $\alpha \in (0, 1.0)$. This is because Padé parameters are in fact functions of both EFT couplings and renormalization scales or constants.

Now, to explore physics using the parametrization given in Eq.(9), we must fix the prescription or Padé parameters through imposing appropriate boundary conditions. To this end, as in most literature, we fit to the PWA[21] data for the phase shifts and mixing angle in the low energy ends, say the kinetic laboratory energy $T_{\text{Lab}} (= 2E) \in (0, 50)\text{MeV}$. In coupled channels, one must fit three sets of Padé parameters for the phase shifts and mixing angle at the same time. In order to see the main points or rationalities of the Padé-aided analysis of coupled channels, we will work with the simplest and hence coarsest cases of Padé approximant, i.e., the constant \mathcal{G} factor,

$$\text{Re}(\mathcal{G}(p))|_{\text{Padé}} \approx \begin{pmatrix} g_{j-1,j-1}^0 & g_{j-1,j+1}^0 \\ g_{j-1,j+1}^0 & g_{j+1,j+1}^0 \end{pmatrix}. \quad (12)$$

Alternatively, with such choice of padé approximant, we wish to probe the most important scales in the nonperturbative factors \mathcal{G} , in order to see if there would be significant deviation from the EFT power counting, or abnormal numbers.

In this short report, we pick up the 3D_3 – 3G_3 channels for illustration where up to next-to-next-to-leading order the potentials contain no extra contact terms to be determined first. This is also the highest coupled channels where at least one channel, 3D_3 , is not perturbative, which could be seen below. As before, we employ the potentials and couplings given by EGM[7]. One could well employ other sets of definitions for comparison. Such works will be carried out in the future. We also note that we deliberately work with low precision in order to save computer workloads: round up to the first two digits.

The numerical results are summarized and presented in TABLE I and Fig.1. The nonperturbative renormalization does significantly improve the phase shift predictions for the 3D_3 channel in comparison with the inferior perturbative ones as depicted in Fig.2, where the perturbative predictions even produce wrong sign of the phase shifts. For δ_{3G_3} we find similar but less significant improvement for lab energies below 100MeV, as the perturbative predictions for all the three orders already deviate from the PWA data from, say $T_{\text{Lab}} \sim 75\text{MeV}$, though less significant. Such simple results already means that only the prescription fixed through physical boundaries could reliably describe physics. (One could try other rather different values of g_{\dots}^0 and see that the phase shifts and mixing angle thus obtained are nonsense.) As the p or energy dependence in $\text{Re}(\mathcal{G})$ is totally discarded here, the predictions are doomed to fail as E is higher, say, $T_{\text{Lab}} > 100\text{MeV}$, which is evident from Fig.1 and Fig.2. Also the trend that the predictions improve order by order is not clear here. To see this trend, more sophisticated Padé approximation and hence heavier workloads are required, which will improve the predictions in many respects. Further works along such lines are in progress and will be reported in the near future. Here, we are merely content with illustrating the plausibility of Padé approximant to the factor $\text{Re}(\mathcal{G})$. Although the Padé approximant adopted here is very coarse, the main virtues in using such relatively more analytical and controllable approach are still quite significant from the simple numerical analysis given in the figures and tables.

We still need to show that the Padé parameters obtained via fitting, here, g_{\dots}^0 , follow the rules described above in Eq.(11) or (10). To this end, we have computed square roots of the absolute values of g_{\dots}^0 and listed them in Table I. From Table I one could see that the scale extracted from the coarse nonperturbative approximation lies between 10 and 200MeV, just in the range described by Eq.(11), that is, $(10, 500)\text{MeV}$. In terms of α , we have $\alpha \in (0.26, 0.91) \subset (0, 1.0)$. In other words, through the Padé approximant of the factor \mathcal{G} , the scales involved in the nonperturbatively renormalized T -matrices do not fall outside of the EFT's scope.

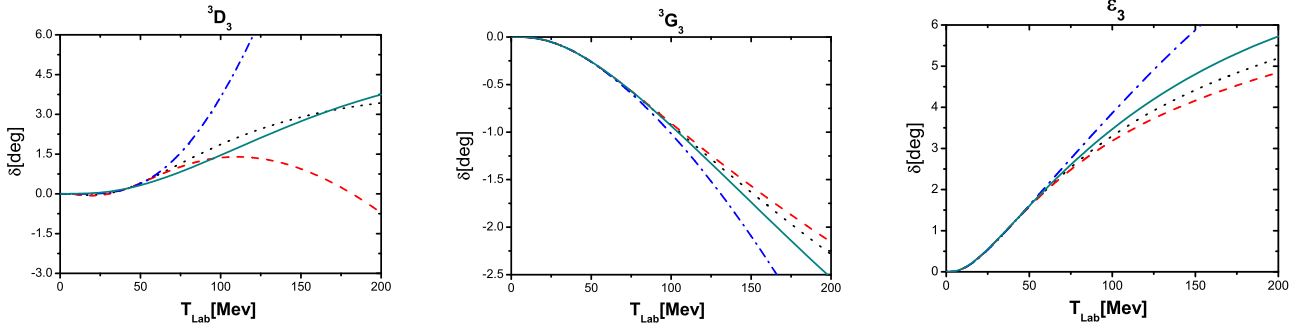
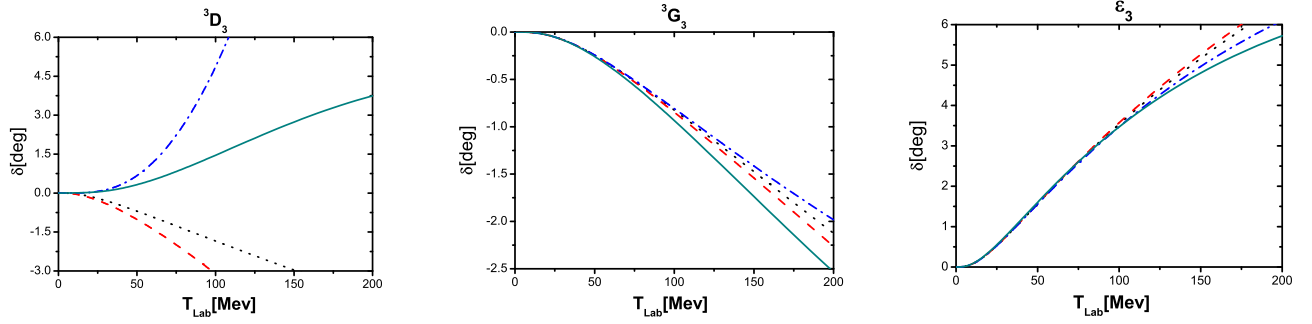
At this stage, we may conclude that the Padé-aided approximation to renormalized T -matrices also works in coupled channels.

In summary, we extended our Padé-aided analysis into coupled channels. Primary numerical analysis showed that such treatment also works in the coupled channels. The results also exhibit that intensive and extensive studies are needed to further develop this promising treatment for investigating various issues in the EFT approach to nucleon-nucleon systems.

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TABLE I: The simplest Padé parameters for $\text{Re}(\mathcal{G})$ ($/\text{MeV}^2$) fitted at different chiral orders.

| | $g_{2,2}^0$ | $\sqrt{ g_{2,2}^0 }$ | $g_{2,4}^0$ | $\sqrt{ g_{2,4}^0 }$ | $g_{4,4}^0$ | $\sqrt{ g_{4,4}^0 }$ |
|------|-------------|----------------------|-------------|----------------------|-------------|----------------------|
| LO | -1300 | 36 | 4200 | 65 | -25000 | 160 |
| NLO | -1500 | 39 | 5200 | 72 | -33000 | 180 |
| NNLO | -770 | 28 | 210 | 14 | 5200 | 72 |

FIG. 1: Predictions of δ_{3D_3} , δ_{3G_3} and ϵ_3 versus lab energy T_{Lab} in MeV with the simplest Padé, with solid line for PWA, dotted lines for LO, dashed line for NLO and dot-dashed line for NNLO.FIG. 2: Perturbative predictions for δ_{3D_3} , δ_{3G_3} and ϵ_3 versus lab energy T_{Lab} in MeV, i.e., $\mathcal{G}=0$. Conventions are as in Fig.1.